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THE TEACHING OF THE CALCULUS.

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Of the various branches of collegiate mathematics doubtless none have received so much attention on the pedagogical side during recent years as the Calculus. In the newer text books on the subject, for example, unusual care has been taken to produce a high order of pedagogical as well as scientific excellence. The same theme has also played an important rôle of late years in the activities of scientific societies, an instance of which occurs in the title of a recent presidential address read before the American Mathematical Society, which was "The Teaching of the Calculus."*

In view of this wide-spread interest in the subject of the present paper, it is with no little hesitation that the author, in accepting an invitation from the Editors of the MONTHLY to express his own views, undertakes to add anything new or of value to it. However, there are two ideas or views which appear to him deserving of somewhat more prominence than they have yet received, and he gladly takes this occasion to at least give expression to them.

Referring again to the newer texts, we may say that they are distinguished from the old mainly in two particulars. First and foremost, they enliven the calculus by bringing it into closer touch with the physical, this in turn being accomplished through the systematic introduction of problems based upon every day experience or taken from the laboratory. Secondly, the logical flaws, chiefly those relating to infinitesimals which were so prevalent in the texts of a generation ago, have been eliminated through the introduction of closer cut definitions of the terms employed. Needless to say, the effect of these betterments has been widely felt and is constantly making its impress in our colleges today. However, there is doubtless room for still further improvement. The present situation may perhaps be described by saying that those pupils, relatively few in number, who really have the power to think may indeed understand the calculus now if they choose, but

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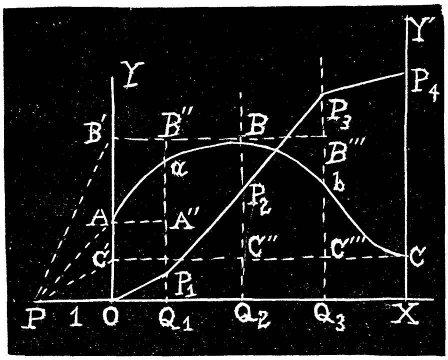
*Address of April 27, 1907, by Professor Wm. F. Osgood. See *Bulletin American Mathematical Society*, Vol. 13, pages 449-467.

that the pupil who has only a fair measure of natural talent still finds it essentially difficult, though somewhat less abtract in its problems than formerly. This may necessarily be so on account of the subtleties of the subject, but I am inclined to the belief that without seriously impairing the dignity or the value of the calculus, it may and should be brought still closer to the comprehension of the average pupil.

In considering how this may be done, I am led to the first idea, mentioned above, which in substance is that the ordinary first course in calculus should be more largely graphical, not in the mere sense of employing frequent geometrical figures to illustrate analytical facts, but in the adoption of some of the ideas belonging properly to the so-called "Graphical Calculus." This latter subject, it may be said, is generally but little known to the pure mathematician, though its importance is of no mean order since it is fast becoming the recognized calculus of the engineer and the physicist. Though of comparatively recent origin, its growth has been very rapid and its methods, whatever else may be said of them, are such as to create a calculus which appeals easily to the average pupil and at the same time has a wide field of usefulness in technical work. It is not my intention to enter into a description of the subject here, since there are abundant sources of information concerning it. However, for the sake of definiteness, I will illustrate what is meant by graphical integration. The particular method followed is one recently brought out by Professor Carl Runge of the University of Goettingen during a course of lectures on graphical methods delivered recently at the University of Michigan.

In the following diagram let it be desired to find the area (given graphically) bounded by the curve ABC, the horizontal line OX and the vertical lines OY, XY.



We first lay off a unit length in the position OP. The point B is then determined by the eye as the one at which the curve has its maximum ordinate. We then determine, again by the eye, the points a, b so that the area $AA''a$=area $aB''B$. Then the required area=area under the rectilinear figure $AA''B''B'''\ C'''\ C$. To find this area graphically, project the points B, C on the line OY, thus obtaining, respectively B', C'. Connect P with $AB'C'$. Then from O draw the line OP_1 parallel to PA, cutting $A''B''$ in P_1. From P_1 draw the line P_1P_2 parallel to PB' cutting BC'' in P_2 and $B'''C'''$ in P_3. Finally from P_3 draw P_3P_4 parallel to PC' cutting XY' in P_4. The line XP_4 represents graphically the desired area.

The correctness of these results is readily established. Thus, in the similar triangles POA, OQ_1P_1 we have $PO:OA :: OQ_1:Q_1P_1$. But $PO=1$ and hence $Q_1P_1=$the area of the rectangle $OQ_1A''A$. Similarly, it may be shown that $Q_3P_3=$the sum of the areas of the rectangles OQ, $A''A$ and $Q_1Q_3B'''B''$, while $XP_4=$the latter sum plus the area of the rectangle Q_3XCC'''.

Greater accuracy may frequently be obtained by dividing each of the arcs AB, BC into several portions (instead of two) such that any one divides the area of its circumscribing rectangle into two equal portions. Again, the selection of a relatively large number of such points and the completion of the subsequent constructions evidently leads to a curve analagous to $OP_1P_2P_3P_4$ but representing a closer approximation to the true integral curve. Thus, the integral curve itself may be sketched in with considerable accuracy, thereby affording a means of computing instantly the area under the curve between any two given ordinates.

We shall not dwell upon the generality of this method of integration or upon various other features of it which are revealed by a close examination. It is to be remarked, however, that many important two dimensional determinations such as areas, centers of gravity, and moments of inertia, are admirably carried out by similar methods. Increased familiarity with the graphical calculus, in fact, always convinces one of its merits, and, as is natural, there are some who would even abandon quite completely the standard form of calculus in favor of the graphical. While it will be seen from what I have to say later on, that such an extreme in no wise seems desirable to me, it may be instructive to note the attitude of the graphical enthusiasts as illustrated in the following quotation taken from the introduction of a recent work.*

"All up to date teachers of engineering and applied science generally now recognize the vast superiority of the graphical over the purely mathematical methods of instruction of almost every description. . . . The attempt to employ purely mathematical, in preference to graphical, methods seems to me quite as absurd as attempting to teach geography by giving the position of towns in terms of their latitude and longitude and explaining the shape of a country by giving the equation of its coast-line instead of by employing the graphical method, that is, exhibiting a map."

The opinions here expressed certainly furnish food for thought. It is undeniably true that the curves with which one deals to-day in applied science are rarely if ever given by means of their equations, but rather graphically. They are determined experimentally by ascertaining a sequence of points satisfying prescribed conditions, and then drawing a smooth curve through them. It is the inability of the ordinary calculus to actually furnish the area under a curve when given in this manner, or to determine centers of gravity and moments of inertia for figures thus bounded—in short, to really *do* things—that is its fatal weakness for the engineer and physicist.

***Graphical Calculus*, by A. H. Barker. London, Longmans, 1905.

Enough has now been said to justify the following inquiry. In view of the recognized importance of the so-called graphical calculus in applied science and in view also of the fact that a large and increasing number of pupils who study the calculus to-day have technical careers in view, would it not be well for the calculus in its present scholastic form to recognize in larger measure the merits of the graphical method and incorporate some of its ideas? My own belief, as already indicated, is that such departures would be advisable. This does not imply an undervaluation of the logical aspect of the calculus but rather a proper valuation of the newer methods and processes which have come to play so important a rôle in it. Neither does it imply a revolution in our methods of instruction, but merely a new point of view in which present conditions, especially as regards the needs of the majority of our pupils, are more squarely met. But to what extent, it may be asked, do you really mean to change the ordinary first course? Would you omit some of the present material in order to make room for the graphical? Doubtless, all would be willing to introduce some of the latter *if there time for it.* Of this I shall have more to say in my second thought, mentioned at the beginning and which I now proceed to consider.

All mathematicians recognize in their scientific work the value of the process of successive approximation. I would now ask whether this process should not be better recognized also on the pedagogical side. In the teaching of the calculus, for example, can we after all expect the average pupil in his first course to appreciate the modern point of view as regards the infinitesimal? True it is that clearness cannot be assured to the scientific mind without such refinements, but should we expect to make the Calculus clear in this sense to a beginner? However heterodox I may appear, I do not believe so. Such insight, in fact, belongs usually to later stages in one's development. The question which the beginner desires answered is, what is the calculus about *about*—what does it *do?* The attitude is the same for any branch of scientific study when first approached and it is only after such questions have been answered that there is a natural inquiry into or need for the really *exact.* Not until then is it time for the further approximations to that end. Moreover, many who are studying the Calculus to-day—in fact the large majority—never reach this second stage at all for lack of natural endowment, yet they have a perfect right to study the subject and are capable of deriving great profit from it. The distinctly higher approximations, leading ultimately into the modern theory of functions of a real variable, are, of course. for those only who possess primarily theoretical inclinations. I believe, therefore, that a more thoughtful application of the successive approximation principle to our teaching would improve it. Likewise, our texts might perhaps be bettered in this particular. They, likewise, should be written in the knowledge that the average beginner is concerned with facts, not with reasons why. Thus it may be said that while our newer texts have indeed brought the calculus nearer home to the average pupil through the introduction of problems based upon daily experience or upon the experiments of the laboratory, still more will stand to the credit of our time when a different arrangement of material has been produced — one, in fact, which will better recognize the successive approximation principle on the pedagogical side. To be more specific, let us be satisfied in the beginning with a rough definition of the derivative, allowing intuition and not rigor to be the chief factor in supplying it and therefore drawing very largely upon the geometrical side. Included in this first approximation should be some simple problems in maxima and minima and the first notions of the integral calculus, all following the same general plan of putting emphasis upon the "what" and *not* the "why." This may be done within the first few weeks of the course after which the second approximation is to be entered upon. In this the whole field should be enlarged and strengthened logically as far as time will permit. It is in this latter connection that opportunity should be found to introduce something at least of the graphical methods as outlined above. Depending upon whether the class is primarily of technical students or otherwise, the teacher should use his judgment as to where the chief emphasis in this second approximation should fall. Conceivably, there are classes in which one might eventually develop the notion of rigor, even in a first course, to such an extent as to acquaint the student with the meaning of an arithmetic *epsilon* proof, while there are classes in which the graphical should largely predominate at the expense of the analytical. Neither element can be omitted, but more judgment should be used in proportioning them.

In recognizing, therefore, the great advance which has been made in later years toward a correct teaching of the Calculus, we would merely point out two features in which still further improvement seems possible. However suicidal it may seem to the analyst, the first course in calculus should be framed to meet more adequately the mental endowments of the average pupil. In particular, we should give a larger recognition to the merits of the so-called graphical calculus. We should also recognize more fully that it is the facts of the calculus and not the logical coherence of it that appeal primarily to the beginner, and this in turn necessitates a fuller recognition on the pedagogical side of the principle of successive approximation.